Assignment 1

This homework is due Friday Jan 24.

There are total 34 points in this assignment. 30 points is considered 100%. If you go over 30 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

(1) (a) [2pt] Show by a picture that $3t_n + t_{n-1} = t_{2n}$ for all $n \geq 2$ (i.e. organize three triangles representing t_n and one triangle representing t_{n-1} into one large triangle).
(b) [2pt] Show the same by direct calculation.
(c) [3pt] (Exercise 2.1.1d in Burton) Show that if n is a triangular number, then so are $9n + 1$, $25n + 3$, $49n + 6$. (By appropriate picture or by a calculation, whichever you prefer.)
(d) [3pt] Generalize the item above. That is, for every odd number $2k+1$, $k \geq 1$, find an integer l such that if n is triangular, then so is

$$(2k + 1)^2 n + l.$$

(*Hint: Direct calculation:* Represent $n = m(m + 1)/2$. Then try to guess which N can provide $t_N = (2k + 1)^2 n + l$. *Picture:* Generalize the picture in the item above.)

(2) (a) [2pt] Express $(2x + 1)^2$ as a linear combination of $1, x, x(x - 1)/2$.
(b) [3pt] Following the proof of in-class theorem, use the above to find a formula for $1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2$.
(c) [3pt] Find the same formula using method of indeterminate coefficients.
(d) [3pt] Re-prove the same using method of mathematical induction.

(3) [3pt] Find a mistake in the following (erroneous!) "proof by mathematical induction":

For any natural n, in any group of n people everyone has the same eye color.

"Proof". If $n = 1$, there is nothing to prove, so we have the base of induction. Suppose that for some fixed n, in any group of n people everyone has the same eye color, and prove the same statement for $n + 1$.

Assume that we have a group of $n + 1$ people $\heartsuit_1, \heartsuit_2, \ldots, \heartsuit_{n+1}$. By the induction hypothesis, n people $\heartsuit_1, \heartsuit_2, \ldots, \heartsuit_n$ have the same eye color. Also, n people $\heartsuit_2, \ldots, \heartsuit_{n+1}$ have the same eye color. Since $\heartsuit_2$ is in both groups, all $n + 1$ people have the same eye color.

(4) [3pt] (1.1.10a) For all $n \geq 1$, prove the following by mathematical induction:

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

— *see next page* —

(5) (a) [3pt] Let A be a matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Prove (for example, by induction) that for all integer $n \geq 1$,

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

(Here F_n is the n-th Fibonacci number:
$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots, F_{n+1} = F_n + F_{n-1}, \ldots$)

(b) [2pt] Find $\det A$; for every integer $n \geq 1$, find $\det A^n$ using properties of determinant.

(c) [2pt] Using results of (a) and (b), prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.